

# The behaviors of expansion functor on monomial ideals and toric rings

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March 13, 2015

## Abstract

In this paper we study some algebraic and combinatorial behaviors of expansion functor. We show that on monomial ideals some properties like polymatroidalness, weakly polymatroidalness and having linear quotients are preserved under taking the expansion functor.

The main part of the paper is devoted to study of toric ideals associated to the expansion of subsets of monomials which are minimal with respect to divisibility. It is shown that, for a given discrete polymatroid  $P$ , if toric ideal of  $P$  is generated by double swaps then toric ideal of any expansion of  $P$  has such a property. This result, in a special case, says that White's conjecture is preserved under taking the expansion functor. Finally, the construction of Gröbner bases and some homological properties of toric ideals associated to expansions of subsets of monomials is investigated.

Keywords: expansion functor, monomial ideal, toric ring, discrete polymatroid, White's conjecture

2010 Mathematics Subject Classifications: 13C13, 13D02.

## Introduction

Let  $S = K[x_1, \dots, x_n]$  be a polynomial ring over a field  $K$  and let  $I$  be a monomial ideal with the set of minimal generators  $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}\}$  where  $\mathbf{x}^{\mathbf{a}_i} = x_1^{\mathbf{a}_i(1)} \dots x_n^{\mathbf{a}_i(n)}$  for  $\mathbf{a}_i = (\mathbf{a}_i(1), \dots, \mathbf{a}_i(n)) \in \mathbb{Z}_+^n = \{\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n : u_i \geq 0\}$ . For the  $n$ -tuple  $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ , Bayati and Herzog [1] defined the expansion of  $I$  with respect to  $\alpha$  in the following form:

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Let  $S^\alpha = K[x_{11}, \dots, x_{1k_1}, \dots, x_{n1}, \dots, x_{nk_n}]$  be a polynomial ring over  $K$  and set  $P_j = (x_{j1}, \dots, x_{jk_j})$  a prime monomial ideal in  $S^\alpha$  for all  $1 \leq j \leq n$ . The expansion of  $I$  with respect to  $\alpha$ , denoted by  $I^\alpha$ , is the monomial ideal

$$I^\alpha = \sum_{i=1}^r P_1^{\mathbf{a}_i(1)} \dots P_n^{\mathbf{a}_i(n)} \subset S^\alpha$$

where  $\mathbf{a}_i(j)$  is the  $j$ -th component of the vector  $\mathbf{a}_i$ .

Define the  $K$ -algebra homomorphism  $\pi : S^\alpha \rightarrow S$  by  $\pi(x_{ij}) = x_i$ .

Let  $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ . For  $\mathbf{u} = (\mathbf{u}(1), \dots, \mathbf{u}(n)) \in \mathbb{Z}_+^n$ , define  $\mathbf{u}^\alpha$  the set of  $|\alpha|$ -tuples  $\mathbf{w} \in \mathbb{Z}^{|\alpha|}$  where  $\mathbf{x}^\mathbf{w} \in G((\mathbf{x}^\mathbf{u})^\alpha)$ . For example, if  $\mathbf{u} = (1, 2, 0)$  and  $\alpha = (2, 2, 2)$  then

$$(\mathbf{x}^\mathbf{u})^\alpha = (x_{11}x_{21}^2, x_{12}x_{21}^2, x_{11}x_{21}x_{22}, x_{11}x_{22}^2, x_{11}x_{21}x_{22}, x_{11}x_{21}x_{22}).$$

Therefore

$$\mathbf{u}^\alpha = \{(1, 0, 2, 0, 0, 0), (0, 1, 1, 1, 0, 0), (1, 0, 1, 1, 0, 0), (1, 0, 0, 2, 0, 0), \\ (1, 0, 1, 1, 0, 0), (1, 0, 1, 1, 0, 0)\}.$$

For  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_+^n$ ,  $\mathbf{u} \preceq \mathbf{v}$  means that  $\mathbf{u}(i) \leq \mathbf{v}(i)$  for all  $i$ . We write  $\mathbf{u} \prec \mathbf{v}$  if  $\mathbf{u} \preceq \mathbf{v}$  and  $\mathbf{u} \neq \mathbf{v}$ . If  $\mathcal{V}$  is a set of vectors in  $\mathbb{Z}_+^n$  which is minimal with respect to  $\preceq$ , then  $\mathcal{V}^\alpha = \bigcup_{\mathbf{u} \in \mathcal{V}} \mathbf{u}^\alpha$ .

Also, for a set  $\mathcal{A}$  of monomials in  $S$  which is minimal with respect to divisibility, we define  $\mathcal{A}^\alpha = \bigcup_{\mathbf{x}^\mathbf{u} \in \mathcal{A}} G((\mathbf{x}^\mathbf{u})^\alpha)$ .

It is easy to see that for a monomial ideal  $I \subset S$ ,  $G(I^\alpha) = \{\mathbf{x}^\mathbf{w} : \mathbf{x}^\mathbf{w} \in G(I)^\alpha\}$ .

In [1] the authors defined the expansion functor in the category of finitely generated multigraded  $S$ -modules and studied some homological behaviors of this functor. In this paper, we consider a subset  $\mathcal{A}$  of monomials which are minimal with respect to divisibility and we study some combinatorial and homological properties on monomial ideals generated by expansions of  $\mathcal{A}$  and also toric ideals related to them. Actually, we investigate some properties on  $K[\mathcal{A}^\alpha]$  or  $I_{\mathcal{A}^\alpha}$  when it holds for  $K[\mathcal{A}]$  or  $I_{\mathcal{A}}$ . The paper is written in two main sections. One section is devoted to study of some behaviors of expansion functor on monomial ideals and the other one is on toric ideals related to expansions of subsets of monomials.

In Section 1, we show that some properties like polymatroidalness (Theorem 1.2), weakly polymatroidalness (Theorem 1.4) and having linear quotients (Theorem 1.7) are preserved under taking the expansion functor.

In Section 2, we discuss several combinatorial and algebraic properties of expansion functor on toric algebras. White [14] conjectured that for a matroid  $\mathcal{M}$ , the toric ideal  $I_{\mathcal{M}}$  is generated by quadrics corresponding to double swaps. On the other hand, matroids are a special subclass of discrete polymatroids, defined in [5]. Herzog and Hibi [5] conjectured that for a discrete polymatroid  $P$ , the toric ideal  $I_P$  is generated by quadrics corresponding to double swaps, too. We will show that for  $\alpha \in \mathbb{N}^n$ , when the toric ideal associated to  $P$  is generated by quadrics corresponding to double swaps then the toric ideal associated to  $P^\alpha$  is (Theorem 2.4). As an application we show that if White's conjecture holds for a matroid  $\mathcal{M}$  then it does for any expansion of  $\mathcal{M}$ . We show that the toric ring  $K[\mathcal{A}^\alpha]$  is normal and Koszul if the toric ring  $K[\mathcal{A}]$  is normal and Koszul. We also conclude that the reduced Gröbner basis of  $I_{\mathcal{A}^\alpha}$  is the intersection of the reduced Gröbner basis of  $I_{\mathcal{A}^\alpha}$  with  $K[\mathcal{A}]$ . In Theorem 2.11, the construction of the reduced Gröbner basis of  $I_{\mathcal{A}^\alpha}$  is described whenever the reduced Gröbner basis of  $I_{\mathcal{A}}$  is given. Then we show that a set of monomials is sortable if and only if its expansion is sortable (Theorem 2.14). Combining this result and a result

due to Sturmfels [13] we conclude that if  $\mathcal{A}$  (resp. its expansion  $\mathcal{A}^\alpha$ ) is sortable then  $I_{\mathcal{A}^\alpha}$  (resp.  $I_{\mathcal{A}}$ ) has a Gröbner basis consisting of the quadratic sorting relations.

We show that the toric ring of a set  $\mathcal{A}$  of monomials is normal if and only if the toric ring of an expansion of  $\mathcal{A}$  is (Theorem 2.16). Finally, we describe some homological relations as Krull dimension, depth, projective dimension and Castelnuovo-Mumford regularity between  $K[\mathcal{A}]$  and  $K[\mathcal{A}^\alpha]$  (Theorem 2.19).

## 1 The expansion of some classes of monomial ideals

In this section we show that the expansion functor has well behavior on monomial ideals with respect to the properties of polymatroidalness, weakly polymatroidalness and having linear quotients. In other words, we prove that a monomial ideal has one of the mentioned properties if and only its expansion has the same property.

**Definition 1.1.** ([5]) A monomial ideal  $I$  of  $S$  is called *polymatroidal* if it satisfies the following conditions:

- (i) all elements in  $G(I)$  have the same degree;
- (ii) if  $u = x_1^{a_1} \dots x_n^{a_n}$  and  $v = x_1^{b_1} \dots x_n^{b_n}$  belong to  $G(I)$  with  $a_i > b_i$ , then there exists  $j$  with  $a_j < b_j$  such that  $x_j(u/x_i) \in G(I)$ .

**Theorem 1.2.** Let  $\alpha \in \mathbb{N}^n$ . The monomial ideal  $I \subset S$  is polymatroidal if and only if  $I^\alpha$  is polymatroidal.

*Proof.* “Only if part”: Let  $I$  be polymatroidal and let  $\alpha = (k_1, \dots, k_n)$ . Let

$$u = x_{11}^{a_{11}} \dots x_{1k_1}^{a_{1k_1}} \dots x_{n1}^{a_{n1}} \dots x_{nk_n}^{a_{nk_n}} \text{ and } v = x_{11}^{b_{11}} \dots x_{1k_1}^{b_{1k_1}} \dots x_{n1}^{b_{n1}} \dots x_{nk_n}^{b_{nk_n}}$$

be two monomials in  $G(I^\alpha)$  with  $a_{ij} > b_{ij}$ . Set  $a_i = \sum_j a_{ij}$  and  $b_i = \sum_j b_{ij}$ . If  $a_i \leq b_i$  then there exists some  $j'$  such that  $a_{ij'} < b_{ij'}$ . Therefore it is clear that  $x_{ij'}(u/x_{ij}) \in G(I^\alpha)$  and the assertion holds. So suppose that  $a_i > b_i$ . Then there exists  $k$  with  $a_k < b_k$  such that  $x_k(\pi(u)/x_i) \in G(I)$ . In particular,  $a_k < b_k$  implies that there is  $t$  with  $a_{kt} < b_{kt}$ . Therefore  $x_{kt}(u/x_{ij}) \in G(I^\alpha)$  which is desired assertion.

“If part”: Let  $I^\alpha$  be polymatroidal and let  $u = x_1^{a_1} \dots x_n^{a_n}$  and  $v = x_1^{b_1} \dots x_n^{b_n}$  with  $a_i > b_i$  be two monomials of  $G(I)$ . Then  $u' = x_{11}^{a_{11}} \dots x_{n1}^{a_{n1}}$  and  $v' = x_{11}^{b_{11}} \dots x_{n1}^{b_{n1}}$  are two monomials in  $G(I^\alpha)$ . Hence there is  $j$  with  $a_j < b_j$  such that  $x_{j1}(u'/x_{i1}) \in G(I^\alpha)$ . Therefore  $x_j(u/x_i) \in G(I)$ .  $\square$

**Definition 1.3.** ([9, 10]) A monomial ideal  $I$  is called *weakly polymatroidal with respect to the ordering*  $x_1 > \dots > x_n$  if for every two monomials  $u = x_1^{a_1} \dots x_n^{a_n}$  and  $v = x_1^{b_1} \dots x_n^{b_n}$  in  $G(I)$  such that  $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$  and  $a_t > b_t$ , there exists  $j > t$  such that  $x_t(v/x_j) \in I$ .

We say that a monomial ideal  $I \subset S$  is weakly polymatroidal if it is weakly polymatroidal with respect to some ordering of variables  $x_1, \dots, x_n$ .

It is clear from the definition that a polymatroidal ideal is weakly polymatroidal but the converse is not true in general (see [9, Example 1.3]).

**Theorem 1.4.** Let  $\alpha \in \mathbb{N}^n$ . The monomial ideal  $I \subset S$  is weakly polymatroidal if and only if  $I^\alpha \subset S^\alpha$  is weakly polymatroidal.

*Proof.* “Only if part”: Suppose that  $I$  is weakly polymatroidal with respect to the ordering  $x_1 > \dots > x_n$ . Let  $\alpha = (k_1, \dots, k_n)$ . We want to show that  $I^\alpha$  is weakly polymatroidal with respect to the ordering  $x_{11} > \dots > x_{1k_1} > \dots > x_{n1} > \dots > x_{nk_n}$ . Let  $u = x_{11}^{a_{11}} \dots x_{1k_1}^{a_{1k_1}} \dots x_{n1}^{a_{n1}} \dots x_{nk_n}^{a_{nk_n}}$  and  $v = x_{11}^{b_{11}} \dots x_{1k_1}^{b_{1k_1}} \dots x_{n1}^{b_{n1}} \dots x_{nk_n}^{b_{nk_n}}$  be two monomials in  $G(I^\alpha)$  such that one of the following properties holds:

- (i)  $a_{11} = b_{11}, \dots, a_{s-1, k_{s-1}} = b_{s-1, k_{s-1}}$  and  $a_{s1} > b_{s1}$ .
- (ii)  $a_{11} = b_{11}, \dots, a_{s, t-1} = b_{s, t-1}$  and  $a_{st} > b_{st}$ .

Set  $a_i = \sum_j a_{ij}$  and  $b_i = \sum_j b_{ij}$ . Suppose (i) holds. We have two cases:

Case 1. Let  $x_s | \pi(v)$ . If  $k_s = 1$ , then there is  $t > s$  such that  $x_s(\pi(v)/x_t) \in I$ . So  $x_{s1}(v/x_{t1}) \in I^\alpha$  for some  $l$ . So assume that  $k_s > 1$ . If  $b_s = 1$ , then there is  $t' > s$  such that  $x_s(\pi(v)/x_{t'}) \in I$  and so  $x_{s1}(v/x_{t'1}) \in I^\alpha$  for some  $l$ . If  $b_s > 1$ , then by the definition of expansion of an ideal there is  $p > 1$  such that  $x_{sp} | v$  and clearly  $x_{s1}(v/x_{sp}) \in I^\alpha$ .

Case 2. Let  $x_s \nmid \pi(v)$ . Since  $I$  is weakly polymatroidal, there is  $k > s$  such that  $x_s(\pi(v)/x_k) \in I$ . Thus  $x_{s1}(v/x_{k1}) \in I^\alpha$  for some  $l$ .

Suppose (ii) holds. If there is  $t' > t$  such that  $x_{st'} | v$ , then it is clear that  $x_{st}(v/x_{st'}) \in I^\alpha$ . Assume  $x_{st'} \nmid v$ , for all  $t' > t$ . Then  $a_s > b_s$  and since  $I$  is weakly polymatroidal we have  $x_s(\pi(v)/x_k) \in I$  for some  $k > s$ . This implies that  $x_{st}(v/x_{kl}) \in I^\alpha$  for some  $l$ .

Therefore  $I^\alpha$  is weakly polymatroidal.

“If part”: Suppose  $I^\alpha$  is weakly polymatroidal with respect to the ordering

$$x_{i_1 j_1} > \dots > x_{i_{|\alpha|} j_{|\alpha|}} \quad (1)$$

which for all  $l$ ,  $1 \leq i_l \leq n$  and  $1 \leq j_l \leq k_{i_l}$ . We will show that  $I$  is weakly polymatroidal with respect to the ordering  $x_{s_1} > \dots > x_{s_n}$  obtained from the ordering (1) after applying the  $K$ -algebra homomorphism  $\pi : S^\alpha \rightarrow S$  by  $\pi(x_{ij}) = x_i$  and removing the repeated variables beginning on the left-hand. In other words, if

$$x_{i_1} \geq \dots \geq x_{i_p} \geq \dots \geq x_{i_q} \geq \dots \geq x_{i_{|\alpha|}}$$

where  $i_p = i_q$  then we will remove  $x_{i_q}$ . Let  $x_{s_l} = \pi(x_{s_l t_l})$  for all  $l = 1, \dots, n$ . Let  $u = x_{s_1}^{a_1} \dots x_{s_n}^{a_n}$ ,  $v = x_{s_1}^{b_1} \dots x_{s_n}^{b_n} \in G(I)$  with  $a_1 = b_1, \dots, a_{j-1} = b_{j-1}$  and  $a_j > b_j$ . Then  $u' = x_{s_1 t_1}^{a_1} \dots x_{s_n t_n}^{a_n} > v' = x_{s_1 t_1}^{a_1} \dots x_{s_n t_n}^{a_n}$  are in  $G(I^\alpha)$  and so there exists  $k > j$  with  $a_k < b_k$  such that  $x_{s_j t_j}(v'/x_{s_k t_k}) \in I^\alpha$ . Thus  $x_{s_j}(v/x_{s_k}) \in I$ . This concludes that  $I$  is weakly polymatroidal.  $\square$

Now we study the behavior of expansion functor on the property of having linear quotients. First, we recall some notations and definitions from [12]:

For the monomial  $u = x_1^{a_1} \dots x_n^{a_n}$  in  $S$ , we will denote the *support* of  $u$  by  $\text{supp}(u)$  and it is the set of those integers  $i$  that  $a_i \neq 0$ . Set  $v_{x_i}(u) := a_i$ . When  $M$  is another monomial, we set  $[u, M] = 1$  if for all  $i \in \text{supp}(u)$ ,  $x_i^{a_i} \nmid v$ . Otherwise we set  $[u, M] \neq 1$ .

For the monomial  $u \in S$  and the monomial ideal  $I \subset S$  set

$$I^u = \langle M_i \in G(I) : [u, M_i] \neq 1 \rangle \quad \text{and} \quad I_u = \langle M_i \in G(I) : [u, M_i] = 1 \rangle.$$

Let the minimal system of generators of  $I$  be  $G(I) = \{M_1, \dots, M_r\}$ . The monomial  $u = x_1^{a_1} \dots x_n^{a_n}$  is called *shedding* if  $I_u \neq 0$  and for each  $M_i \in G(I_u)$  and each  $l \in \text{supp}(u)$  there exists  $M_j \in G(I^u)$  such that  $M_j : M_i = x_l$ .

**Definition 1.5.** ([12]) Let  $I$  be a monomial ideal minimally generated by  $\{M_1, \dots, M_r\}$ . We say  $I$  is a  $k$ -decomposable ideal if  $r = 1$  or else has a shedding monomial  $u$  with  $|\text{supp}(u)| \leq k + 1$  such that the ideals  $I^u$  and  $I_u$  are  $k$ -decomposable.

For each monomial ideal  $I$  and a system of minimal generators  $u_1, \dots, u_r$  of  $I$ , we say that  $I$  has *linear quotients with respect to the ordering*  $u_1, \dots, u_r$  if for all  $j = 2, \dots, r$ , the colon ideal  $(u_1, \dots, u_{j-1}) : (u_j)$  is generated by linear forms. In other words, if for all  $j < i$  there exists an integer  $k < i$  and an integer  $l$  such that

$$\frac{u_k}{\gcd(u_k, u_i)} = x_l \mid \frac{u_j}{\gcd(u_j, u_i)}.$$

A monomial ideal which has linear quotients with respect to some ordering of minimal generators, is called a monomial ideal with linear quotients.

It is known that a weakly polymatroidal ideal has linear quotients. In [12] the authors proved that

**Theorem 1.6.** ([12, Theorem 2.13]) *A monomial ideal has linear quotients if and only if it is  $k$ -decomposable for some  $k \geq 0$ .*

We will use this theorem in the following.

**Theorem 1.7.** *Let  $\alpha \in \mathbb{N}^n$ . The monomial ideal  $I \subset S$  has linear quotients if and only if  $I^\alpha \subset S^\alpha$  has linear quotients.*

*Proof.* “Only if part”: We use induction on the number of minimal generators of  $I$ . Since  $I$  has linear quotients, so there exists a shedding monomial  $u = x_{i_1}^{a_1} \dots x_{i_m}^{a_m}$  for  $I$  such that  $I^u$  and  $I_u$  have linear quotients, by Theorem 1.6. Set  $J = I^\alpha$  and

$$J^0 = (M \in G(J) \mid [u, \pi(M)] \neq 1) \text{ and } J_0 = (M \in G(J) \mid [u, \pi(M)] = 1).$$

Note that  $J^0 = (I^u)^\alpha$  and  $J_0 = (I_u)^\alpha$ . Let  $J^0$  and  $J_0$  are, respectively, minimally generated by  $M_1, \dots, M_r$  and  $M_{r+1}, \dots, M_s$  and moreover, they have linear quotients with respect the given orderings. We want to show that  $J$  has linear quotients with respect to  $M_1, \dots, M_s$ .

Let  $M_p$  and  $M_q$  are in  $G(J)$  with  $p < q$ . Set  $N_i := \pi(M_i)$ . By the induction hypothesis,  $J^0$  and  $J_0$  have linear quotients. Thus it suffices to consider that  $p \leq r < q$ . It is clear that  $N_p \in G(I^u)$  and  $N_q \in G(I_u)$ . Therefore there is  $t \leq p$  such that  $N_t / \gcd(N_t, N_q) = x_{i_l}$  for some  $1 \leq l \leq m$  and  $x_{i_l}$  divides  $N_p / \gcd(N_p, N_q)$ . It concludes that  $M_p / \gcd(M_p, M_q)$  is divided by  $x_{i_l h}$  for some  $1 \leq h \leq k_{i_l}$ . To complete the assertion, one can choose a monomial  $M_{t'} \in \pi^{-1}(N_t)$  with the property  $M_{t'} / \gcd(M_{t'}, M_q) = x_{i_l h}$ . Clearly,  $t' \leq p$ .

“If part”: Let  $I^\alpha$  has linear quotients with respect to the ordering  $M_1, \dots, M_s$ . Set  $N_l := \pi(M_l)$ . Suppose that we obtain the ordering  $N_{i_1}, \dots, N_{i_r}$ , with disjoint monomials, after removing any repeated monomial beginning on the left-hand of the ordering  $N_1, \dots, N_s$ . We want to show that  $I$  has linear quotients with respect to  $N_{i_1}, \dots, N_{i_r}$ .

Consider two monomials  $N_{i_p}$  and  $N_{i_q}$  with  $p < q$ . Let  $N_{i_p} = \pi(M_{p'})$  and  $N_{i_q} = \pi(M_{q'})$ . Clearly,  $p' < q'$ . Therefore there exist  $k' < q'$  and a variable  $x_{lm}$  such that  $M_{k'} / \gcd(M_{k'}, M_{q'}) = x_{lm}$  and, moreover,  $x_{lm}$  divides  $M_{p'} / \gcd(M_{p'}, M_{q'})$ . Let  $N_{i_k} = \pi(M_{k'})$ . Since that  $N_{i_k} \nmid N_{i_q}$ , we have  $\gcd(N_{i_k}, N_{i_q}) \neq N_{i_k}$ . Therefore  $N_{i_k} / \gcd(N_{i_k}, N_{i_q}) \neq 1$ . Now let  $x_t^a$  divide  $N_{i_k} / \gcd(N_{i_k}, N_{i_q})$ . Then for some  $s$ ,  $x_{ts}$  divides  $M_{k'} / \gcd(M_{k'}, M_{q'})$ . This implies that  $a = 1$ ,  $t = l$  and  $s = m$ . Therefore  $N_{i_k} / \gcd(N_{i_k}, N_{i_q}) = x_l$ . Similarly, we show that  $x_l$  divides  $N_{i_p} / \gcd(N_{i_p}, N_{i_q})$ , as desired.  $\square$

**Remark 1.8.** The only if part of Theorem 1.7 was proved, in a different argument, in Proposition 1.6 of [1].

**Remark 1.9.** For a given  $\alpha \in \mathbb{N}^n$  and a monomial ideal  $I \subset S$  generated in one degree, we can abbreviate Theorems 1.2, 1.4, 1.7 and also Theorem 4.2 of [1] in the following implications:

$$\begin{array}{ccc}
I \text{ is polymatroidal} & \Leftrightarrow & I^\alpha \text{ is polymatroidal} \\
\Downarrow & & \Downarrow \\
I \text{ is weakly polymatroidal} & \Leftrightarrow & I^\alpha \text{ is weakly polymatroidal} \\
\Downarrow & & \Downarrow \\
I \text{ has linear quotients} & \Leftrightarrow & I^\alpha \text{ has linear quotients} \\
\Downarrow & & \Downarrow \\
I \text{ has linear resolution} & \Leftrightarrow & I^\alpha \text{ has linear resolution}
\end{array}$$

## 2 The expansion functor and toric algebra

### 2.1 White's conjecture

White [14] conjectured that for a matroid  $\mathcal{M}$ , the toric ideal associated to  $\mathcal{M}$ ,  $I_{\mathcal{M}}$ , is generated by quadrics corresponding to double swaps. In the previous section it was shown that a monomial ideal is polymatroidal if and only if its expansion is polymatroidal. Since that a polymatroidal ideal is generated by monomials corresponding to the base of a discrete polymatroid, and also since matroids are a special subclass of discrete polymatroids, it is then natural to ask about holding white's conjecture for expansion of a discrete polymatroid. We first bring some notations and definitions.

**Definition 2.1.** ([5]) A *discrete polymatroid* on the ground set  $[n]$  is a nonempty finite set  $P \subset \mathbb{Z}_+^n$  satisfying

(D1) if  $\mathbf{v} \in \mathbb{Z}_+^n$  with  $\mathbf{v} \preceq \mathbf{u}$  for some  $\mathbf{u} \in P$ , then  $\mathbf{v} \in P$ ;

(D2) if  $\mathbf{u}, \mathbf{v} \in P$  with  $|\mathbf{u}| < |\mathbf{v}|$ , then there is  $i \in [n]$  with  $\mathbf{u}(i) < \mathbf{v}(i)$  such that  $\mathbf{u} + \varepsilon_i \in P$ .

Here  $\varepsilon_i$  denotes the  $i$ th canonical basis vector in  $\mathbb{R}^n$ .

A *base* of  $P$  is a vector  $\mathbf{u} \in P$  such that  $\mathbf{u} \prec \mathbf{v}$  for no  $\mathbf{v} \in P$ . It follows from (D1) and (D2) that a nonempty finite set  $B \subset \mathbb{Z}_+^n$  is the set of bases of a discrete polymatroid on  $[n]$  if and only if  $B$  satisfies the following conditions:

- (i) all elements of  $B$  have the same modulus;
- (ii) if  $\mathbf{u}, \mathbf{v} \in P$  belong to  $B$  with  $\mathbf{u}(i) > \mathbf{v}(i)$ , then there is  $j \in [n]$  with  $\mathbf{u}_j < \mathbf{v}_j$  such that  $\mathbf{u} - \varepsilon_i + \varepsilon_j \in B$ .

We will denote by  $\mathcal{B}_P$  the set of bases of  $P$  on  $[n]$ .

Let  $P \subset \mathbb{Z}_+^n$  be a discrete polymatroid and  $\mathcal{B}_P$  its set of bases. Define  $S_P = K[y_{\mathbf{u}} : \mathbf{u} \in \mathcal{B}_P]$  a polynomial ring over  $K$  and write  $I_P \subset S_P$  for the toric ideal of the *base ring*  $K[P] := K[\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in \mathcal{B}_P]$  where  $\mathbf{x}^{\mathbf{u}} = x_1^{\mathbf{u}(1)} \dots x_n^{\mathbf{u}(n)}$  for  $\mathbf{u} = (\mathbf{u}(1), \dots, \mathbf{u}(n)) \in \mathbb{Z}_+^n$ . In other words,  $I_P$  is the kernel of the surjective  $K$ -algebra homomorphism  $\varphi_P : S_P \rightarrow K[P]$  defined by  $\varphi_P(y_{\mathbf{u}}) = \mathbf{x}^{\mathbf{u}}$ .

We say that a pair of bases  $(\mathbf{v}_1, \mathbf{v}_2)$  is obtained from a pair of bases  $(\mathbf{u}_1, \mathbf{u}_2)$  by a *double swap* if  $\mathbf{v}_1 = \mathbf{u}_1 + \varepsilon_j - \varepsilon_i$  and  $\mathbf{v}_2 = \mathbf{u}_2 + \varepsilon_i - \varepsilon_j$  for some  $i, j$  with  $\mathbf{u}_1(i) > \mathbf{u}_2(i)$  and  $\mathbf{u}_2(j) > \mathbf{u}_1(j)$ . In this case we write  $(\mathbf{v}_1, \mathbf{v}_2) \sim_P (\mathbf{u}_1, \mathbf{u}_2)$ .

Recall that for the canonical basis vector  $\varepsilon_i \in \mathbb{R}^n$ ,

$$\varepsilon_i^\alpha = \{\varepsilon_{i1}, \dots, \varepsilon_{ik_i}\}$$

which  $\varepsilon_{ij}$  is a canonical basis vector of  $\mathbb{R}^{|\alpha|}$  with the  $(k_1 + \dots + k_{i-1} + j)$ -th component equal to 1 and zero for other components.

For a discrete polymatroid  $P \subset \mathbb{Z}_+^n$ ,  $P^\alpha \subset \mathbb{Z}^{|\alpha|}$  is defined a discrete polymatroid which its set of bases is  $\mathcal{B}_{P^\alpha} = \bigcup_{\mathbf{u} \in \mathcal{B}_P} \mathbf{u}^\alpha$ .

**Example 2.2.** Consider the discrete polymatroid  $P$  with the singleton base set  $\mathcal{B}_P = \{\mathbf{u} := (1, 1)\}$  and let  $\alpha = (2, 2)$ . Then

$$\mathcal{B}_{P^\alpha} = \{\mathbf{u}_1 := (1, 0, 1, 0), \mathbf{u}_2 := (1, 0, 0, 1), \mathbf{u}_3 := (0, 1, 1, 0), \mathbf{u}_4 := (0, 1, 0, 1)\}.$$

Moreover,

$$I_P = 0 \text{ and } I_{P^\alpha} = (y_{\mathbf{u}_1}y_{\mathbf{u}_4} - y_{\mathbf{u}_2}y_{\mathbf{u}_3}).$$

Define the surjective map

$$\pi_0 : \mathbb{Z}^{|\alpha|} \rightarrow \mathbb{Z}^n$$

by  $\pi_0(\mathbf{u}) = (a_1, \dots, a_n)$  for  $\mathbf{u} = (a_{11}, \dots, a_{1k_1}, \dots, a_{n1}, \dots, a_{nk_n}) \in \mathbb{Z}^{|\alpha|}$  where  $a_j = \sum_{l=1}^{k_j} a_{jl}$  for all  $j$ . We also define the  $K$ -algebra epimorphism

$$\begin{aligned} \tau : K[P^\alpha] &\rightarrow K[P] \\ \mathbf{x}^{\mathbf{u}} &\mapsto \mathbf{x}^{\pi_0(\mathbf{u})}. \end{aligned}$$

Also, the  $K$ -algebra homomorphism  $\gamma : S_{P^\alpha} \rightarrow S_P$  is defined by  $\gamma(y_{\mathbf{u}}) = y_{\pi_0(\mathbf{u})}$ . Therefore we have the following commutative diagram from surjective maps:

$$\begin{array}{ccc} S_{P^\alpha} & \xrightarrow{\varphi_{P^\alpha}} & K[P^\alpha] \\ \gamma \downarrow & & \downarrow \tau \\ S_P & \xrightarrow{\varphi_P} & K[P]. \end{array}$$

Before proving the main theorem of this subsection we require the following lemmas.

**Lemma 2.3.** For  $\alpha \in \mathbb{N}^n$ ,  $\gamma(I_{P^\alpha}) = I_P$ .

*Proof.* Let  $f \in I_{P^\alpha}$ . Then  $\varphi_P(\gamma(f)) = \tau(\varphi_{P^\alpha}(f)) = 0$  and so  $\gamma(f) \in I_P$ . For the converse inclusion, let  $g = y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} \in I_P$ . Then for all  $i$ , set  $\mathbf{u}'_i := \sum_j \mathbf{u}_i(j) \varepsilon_{j1}$ ,  $\mathbf{v}'_i := \sum_j \mathbf{v}_i(j) \varepsilon_{j1}$  and  $h := y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}$ . Hence  $\gamma(h) = g$ . By the fact that a binomial  $y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} \in I_P$  if and only if  $\sum^m \mathbf{u}_i = \sum^m \mathbf{v}_i$ , we conclude that  $\sum^m \mathbf{u}'_i = \sum^m \mathbf{v}'_i$ . Therefore  $h \in I_{P^\alpha}$  and so  $g \in \gamma(I_{P^\alpha})$ .  $\square$

**Theorem 2.4.** Let  $\alpha \in \mathbb{N}^n$  and  $P$  be a discrete polymatroid. If  $I_P$  is generated by quadratic binomials corresponding to double swaps then  $I_{P^\alpha}$  is, too.

*Proof.* Suppose  $I_P$  is generated by quadratic binomials corresponding to double swaps. We consider a binomial  $f \in I_{P^\alpha}$  and show that we can write  $f$  as sum of quadratic binomials corresponding to double swaps.

Suppose  $0 \neq f \in I_{P^\alpha}$  is of degree 2. Let  $f = y_{\mathbf{u}'_1}y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1}y_{\mathbf{v}'_2}$ . Let  $\pi_0(\mathbf{u}'_i) = \mathbf{u}_i$  and  $\pi_0(\mathbf{v}'_i) = \mathbf{v}_i$ . By Lemma 2.3,  $\gamma(y_{\mathbf{u}'_1}y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1}y_{\mathbf{v}'_2}) = y_{\mathbf{u}_1}y_{\mathbf{u}_2} - y_{\mathbf{v}_1}y_{\mathbf{v}_2} \in I_P$ . If  $y_{\mathbf{u}_1}y_{\mathbf{u}_2} - y_{\mathbf{v}_1}y_{\mathbf{v}_2} = 0$ , then one can assume that  $\mathbf{u}_1 = \mathbf{v}_1$  and  $\mathbf{u}_2 = \mathbf{v}_2$ . It follows from  $\mathbf{u}'_1 + \mathbf{u}'_2 = \mathbf{v}'_1 + \mathbf{v}'_2$  that

$$\mathbf{v}'_1 = \mathbf{u}'_1 + \mathbf{w}' - \mathbf{z}', \quad \mathbf{v}'_2 = \mathbf{u}'_2 + \mathbf{z}' - \mathbf{w}'$$

where  $\mathbf{w}' = \sum_{i=f_1}^{f_r} \sum_{j=g_1}^{g_s} t_{ij} \varepsilon_{ij}$ ,  $\mathbf{z}' = \sum_{i=f_1}^{f_r} \sum_{j=h_1}^{h_t} s_{ij} \varepsilon_{ij}$  and, moreover,  $\sum_j t_{ij} = \sum_j s_{ij}$ . Therefore we can write  $y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1} y_{\mathbf{v}'_2}$  as a sum of quadratic binomials corresponding to double swaps:

$$y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1} y_{\mathbf{v}'_2} = (y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{u}'_1 + \varepsilon_{f_1 g_1} - \varepsilon_{f_1 h_1}} y_{\mathbf{u}'_2 + \varepsilon_{f_1 h_1} - \varepsilon_{f_1 g_1}}) + \dots + (y_{\mathbf{u}'_1 + \sum_{i=f_1}^{f_r} \sum_{j=g_1}^{g_s-1} t_{ij} \varepsilon_{ij} - \sum_{i=f_1}^{f_r} \sum_{j=h_1}^{h_t-1} s_{ij} \varepsilon_{ij}} y_{\mathbf{u}'_2 + \sum_{i=f_1}^{f_r} \sum_{j=h_1}^{h_t-1} s_{ij} \varepsilon_{ij} - \sum_{i=f_1}^{f_r} \sum_{j=g_1}^{g_s-1} t_{ij} \varepsilon_{ij}} - y_{\mathbf{v}'_1} y_{\mathbf{v}'_2}).$$

Thus suppose that  $y_{\mathbf{u}_1} y_{\mathbf{u}_2} - y_{\mathbf{v}_1} y_{\mathbf{v}_2} \neq 0$ . By the assumption, we have

$$y_{\mathbf{u}_1} y_{\mathbf{u}_2} - y_{\mathbf{v}_1} y_{\mathbf{v}_2} = \sum_{i=1}^t (y_{\mathbf{u}_{i1}} y_{\mathbf{u}_{i2}} - y_{\mathbf{v}_{i1}} y_{\mathbf{v}_{i2}})$$

where  $(\mathbf{v}_{i1}, \mathbf{v}_{i2}) \sim_P (\mathbf{u}_{i1}, \mathbf{u}_{i2})$  for all  $i$ .

**Case 1:** If  $t = 1$ , then  $(\mathbf{v}_1, \mathbf{v}_2) \sim_P (\mathbf{u}_1, \mathbf{u}_2)$ . Let  $\mathbf{v}_1 = \mathbf{u}_1 + \varepsilon_{i_q} - \varepsilon_{i_p}$  and  $\mathbf{v}_2 = \mathbf{u}_2 + \varepsilon_{i_p} - \varepsilon_{i_q}$ .

Then we will have  $\mathbf{v}'_1 = \mathbf{u}'_1 + \varepsilon_{i_q j_{q'}} - \varepsilon_{i_p j_{p'}}$  and  $\mathbf{v}'_2 = \mathbf{u}'_2 + \varepsilon_{i_p j_{p'}} - \varepsilon_{i_q j_{q'}}$ . Now  $\mathbf{u}'_1 + \mathbf{u}'_2 = \mathbf{v}'_1 + \mathbf{v}'_2$  implies that  $j_{p'} = j_{p''}$  and  $j_{q'} = j_{q''}$ . Thus  $(\mathbf{v}'_1, \mathbf{v}'_2) \sim_{P^\alpha} (\mathbf{u}'_1, \mathbf{u}'_2)$ .

**Case 2:** If  $t > 1$ , then by choosing suitable  $\mathbf{u}'_{ij}$ 's from  $\mathcal{B}_{P^\alpha}$  one may consider

$$y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1} y_{\mathbf{v}'_2} = \sum_{i=1}^t (y_{\mathbf{u}'_{i1}} y_{\mathbf{u}'_{i2}} - y_{\mathbf{v}'_{i1}} y_{\mathbf{v}'_{i2}})$$

where  $\pi_0(\mathbf{u}'_{ij}) = \mathbf{u}_{ij}$  and  $\pi_0(\mathbf{v}'_{ij}) = \mathbf{v}_{ij}$  for all  $i$  and  $j$  and, moreover,  $y_{\mathbf{u}'_{i1}} y_{\mathbf{u}'_{i2}} - y_{\mathbf{v}'_{i1}} y_{\mathbf{v}'_{i2}} \in I_{P^\alpha}$ . It follows from case 1 that  $y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{v}'_1} y_{\mathbf{v}'_2}$  is generated by quadratic binomials corresponding to double swaps.

Now suppose that  $f = y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} \in I_{P^\alpha}$  where  $m > 2$  and every binomial  $y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_{m'}} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_{m'}} \in I_{P^\alpha}$  with  $m' < m$  is generated by quadratic binomials corresponding to double swaps. If  $\mathbf{u}'_i = \mathbf{v}'_j$  for some  $1 \leq i, j \leq m$  then

$$y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} = y_{\mathbf{u}'_i} (y_{\mathbf{u}'_1} \dots \hat{y}_{\mathbf{u}'_i} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots \hat{y}_{\mathbf{v}'_j} \dots y_{\mathbf{v}'_m})$$

which is generated by quadratic binomials corresponding to double swaps, by induction hypothesis. So assume that  $\mathbf{u}'_i \neq \mathbf{v}'_j$  for all  $1 \leq i, j \leq m$ . Let  $\mathbf{u}_i = \pi_0(\mathbf{u}'_i)$  and  $\mathbf{v}_i = \pi_0(\mathbf{v}'_i)$ , for all  $i$ . It follows from  $\sum^m \mathbf{u}_i = \sum^m \mathbf{v}_i$  that  $y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} \in I_P$ . First, let  $y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} = 0$ . Then we can suppose that  $\mathbf{u}_1 = \mathbf{v}_1$ . Let  $\mathbf{u}'_1(ij) > \mathbf{v}'_1(ij)$  and  $\mathbf{u}'_1(ik) < \mathbf{v}'_1(ik)$ . This implies that there is  $\mathbf{u}'_l$  with  $\mathbf{u}'_l(ik) > 0$ . For convenience, let us set  $l = 2$ . Hence we can write

$$\begin{aligned} y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} &= (y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} \dots y_{\mathbf{u}'_m} - y_{\mathbf{u}'_1 - \varepsilon_{ij} + \varepsilon_{ik}} y_{\mathbf{u}'_2 - \varepsilon_{ik} + \varepsilon_{ij}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m}) + \\ &\quad (y_{\mathbf{u}'_1 - \varepsilon_{ij} + \varepsilon_{ik}} y_{\mathbf{u}'_2 - \varepsilon_{ik} + \varepsilon_{ij}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}) \\ &= (y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{u}'_1 - \varepsilon_{ij} + \varepsilon_{ik}} y_{\mathbf{u}'_2 - \varepsilon_{ik} + \varepsilon_{ij}}) y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} + \\ &\quad (y_{\mathbf{u}'_1 - \varepsilon_{ij} + \varepsilon_{ik}} y_{\mathbf{u}'_2 - \varepsilon_{ik} + \varepsilon_{ij}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}). \end{aligned}$$

Set  $\mathbf{u}''_1 = \mathbf{u}'_1 - \varepsilon_{ij} + \varepsilon_{ik}$  and  $\mathbf{u}''_2 = \mathbf{u}'_2 - \varepsilon_{ik} + \varepsilon_{ij}$ . Clearly,  $y_{\mathbf{u}''_1} y_{\mathbf{u}''_2} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} \in I_{P^\alpha}$  and  $\pi_0(\mathbf{u}''_1) = \pi_0(\mathbf{v}'_1)$ . Now we repeat the above procedure for  $y_{\mathbf{u}''_1} y_{\mathbf{u}''_2} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}$  and after a finite number of steps, we obtain

$$y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} = \sum f_i g_i + y_{\mathbf{v}'_1} (y_{\mathbf{u}''_2} y_{\mathbf{u}''_3} \dots y_{\mathbf{u}''_m} - y_{\mathbf{v}'_2} \dots y_{\mathbf{v}'_m})$$

where  $f_i$ 's are monomials in  $S_{P^\alpha}$  and  $g_i$ 's are quadratic binomials corresponding to double swaps. By induction hypothesis,  $y_{\mathbf{u}''_2} y_{\mathbf{u}''_3} \dots y_{\mathbf{u}''_m} - y_{\mathbf{v}'_2} \dots y_{\mathbf{v}'_m} \in I_{P^\alpha}$  is generated by quadratic



binomials corresponding to double swaps. Therefore the assertion holds when  $y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} = 0$ . So assume that  $y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} \neq 0$ . Therefore, by assumption,

$$y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} = \sum_{i=1}^t f_i (y_{\mathbf{u}_{i1}} y_{\mathbf{u}_{i2}} - y_{\mathbf{v}_{i1}} y_{\mathbf{v}_{i2}})$$

where  $(\mathbf{v}_{i1}, \mathbf{v}_{i2}) \sim_P (\mathbf{u}_{i1}, \mathbf{u}_{i2})$  and  $f_i$ 's are monomials in  $S_P$ . Without loss of generality we may assume that

$$y_{\mathbf{u}_1} \dots y_{\mathbf{u}_m} - y_{\mathbf{v}_1} \dots y_{\mathbf{v}_m} = \sum_{i=1}^t (y_{\mathbf{u}_{i1}} y_{\mathbf{u}_{i2}} y_{\mathbf{u}_{i3}} \dots y_{\mathbf{u}_{im}} - y_{\mathbf{v}_{i1}} y_{\mathbf{v}_{i2}} y_{\mathbf{u}_{i3}} \dots y_{\mathbf{u}_{im}}) \quad (2)$$

which for every  $i$ ,  $(\mathbf{v}_{i1}, \mathbf{v}_{i2}) \sim_P (\mathbf{u}_{i1}, \mathbf{u}_{i2})$ . Clearly, for all  $i$ ,  $y_{\mathbf{u}_{i1}} y_{\mathbf{u}_{i2}} y_{\mathbf{u}_{i3}} \dots y_{\mathbf{u}_{im}} - y_{\mathbf{v}_{i1}} y_{\mathbf{v}_{i2}} y_{\mathbf{u}_{i3}} \dots y_{\mathbf{u}_{im}} \in I_P$ .

**Case 1':** If  $t = 1$ , then we may assume that  $\mathbf{u}_{1j} = \mathbf{u}_j$  for all  $j = 1, \dots, m$ ,  $\mathbf{v}_{1j} = \mathbf{v}_j$  for  $j = 1, 2$  and  $\mathbf{v}_{1j} = \mathbf{u}_j$  for all  $j = 3, \dots, m$ . Let  $\mathbf{v}_1 = \mathbf{u}_1 - \varepsilon_i + \varepsilon_j$  and  $\mathbf{v}_2 = \mathbf{u}_2 - \varepsilon_j + \varepsilon_i$ . Since that  $\mathbf{u}_1(i) > \mathbf{u}_2(i)$  and  $\mathbf{u}_1(j) < \mathbf{u}_2(j)$ , it follows that there are  $r$  and  $s$  such that  $\mathbf{u}'_1(ir) > \mathbf{u}'_2(ir)$  and  $\mathbf{u}'_1(js) < \mathbf{u}'_2(js)$ . We have

$$\begin{aligned} y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} &= (y_{\mathbf{u}'_1} y_{\mathbf{u}'_2} - y_{\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js}} y_{\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir}}) y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} \\ &\quad + y_{\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js}} y_{\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} \end{aligned}$$

Note that  $\pi_0(\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js}) = \pi_0(\mathbf{v}'_1)$  and  $\pi_0(\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir}) = \pi_0(\mathbf{v}'_2)$  and  $y_{\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js}} y_{\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} \in I_P^\alpha$ . Since  $y_{\pi_0(\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js})} y_{\pi_0(\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir})} y_{\pi_0(\mathbf{u}'_3)} \dots y_{\pi_0(\mathbf{u}'_m)} - y_{\pi_0(\mathbf{v}'_1)} \dots y_{\pi_0(\mathbf{v}'_m)} = 0$ , it follows from case 2 that  $y_{\mathbf{u}'_1 - \varepsilon_{ir} + \varepsilon_{js}} y_{\mathbf{u}'_2 - \varepsilon_{js} + \varepsilon_{ir}} y_{\mathbf{u}'_3} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}$  is generated by quadratic binomials corresponding to double swaps and so  $y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}$  is generated by quadratic binomials corresponding to double swaps.

**Case 2':** If  $t > 1$ , then considering the equality (2), we can choose some suitable bases  $\mathbf{u}'_{ij}$  and  $\mathbf{v}'_{ij}$  of  $\mathcal{B}^{P\alpha}$ , with  $\pi_0(\mathbf{u}'_{ij}) = \mathbf{u}_{ij}$  and  $\pi_0(\mathbf{v}'_{ij}) = \mathbf{v}_{ij}$  such that

$$y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m} = \sum_{i=1}^t (y_{\mathbf{u}'_{i1}} y_{\mathbf{u}'_{i2}} y_{\mathbf{u}'_{i3}} \dots y_{\mathbf{u}'_{im}} - y_{\mathbf{v}'_{i1}} y_{\mathbf{v}'_{i2}} y_{\mathbf{v}'_{i3}} \dots y_{\mathbf{v}'_{im}})$$

and  $y_{\mathbf{u}'_{i1}} y_{\mathbf{u}'_{i2}} y_{\mathbf{u}'_{i3}} \dots y_{\mathbf{u}'_{im}} - y_{\mathbf{v}'_{i1}} y_{\mathbf{v}'_{i2}} y_{\mathbf{v}'_{i3}} \dots y_{\mathbf{v}'_{im}} \in I_P^\alpha$ . Now by the case 1', every binomial  $y_{\mathbf{u}'_{i1}} y_{\mathbf{u}'_{i2}} y_{\mathbf{u}'_{i3}} \dots y_{\mathbf{u}'_{im}} - y_{\mathbf{v}'_{i1}} y_{\mathbf{v}'_{i2}} y_{\mathbf{v}'_{i3}} \dots y_{\mathbf{v}'_{im}}$  is generated by quadratic binomials corresponding to double swaps and so  $y_{\mathbf{u}'_1} \dots y_{\mathbf{u}'_m} - y_{\mathbf{v}'_1} \dots y_{\mathbf{v}'_m}$  has the same property, as desired.  $\square$

Since a matroid may be regarded as a discrete polymatroid with the set of bases consisting of  $(0, 1)$ -vectors, so we can conclude the following.

**Corollary 2.5.** *Let  $\alpha \in \mathbb{N}^n$  and  $\mathcal{M}$  be a matroid. If  $\mathcal{M}$  satisfies the White's conjecture then  $\mathcal{M}^\alpha$  does, too.*

**Remark 2.6.** We conjecture that the converse of Theorem 2.4 holds but we could not present any proof. Indeed, this will conclude the converse of Corollary 2.5.

## 2.2 Some combinatorial and algebraic properties

Let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$  and suppose that the affine semigroup ring  $K[\mathcal{A}] = K[u_1, \dots, u_m]$  is a homogeneous  $K$ -algebra. Let

$S_{\mathcal{A}} := K[y_{u_1}, \dots, y_{u_m}]$  be the polynomial ring in  $n$  variables over  $K$  with each  $\deg(y_{u_i}) = 1$  and let  $I_{\mathcal{A}}$  denote the kernel of the surjective homomorphism  $\varphi_{\mathcal{A}} : S_{\mathcal{A}} \rightarrow K[\mathcal{A}]$  defined by  $\varphi_{\mathcal{A}}(y_{u_i}) = u_i$  for all  $1 \leq i \leq m$ .  $I_{\mathcal{A}}$  and  $K[\mathcal{A}]$  are, respectively, called *toric ideal* and *toric ring* of  $\mathcal{A}$ .

Firstly, we recall the concept of *combinatorial pure subring* of a toric ring, introduced in [11], which we will use in the rest of the paper. Let  $T \subseteq [n] := \{x_1, \dots, x_n\}$ . If  $T$  is a nonempty subset of  $[n]$ , then we set  $\mathcal{A}_T := \mathcal{A} \cap K[\{x_i : x_i \in T\}]$ . A subring of  $K[\mathcal{A}]$  of the form  $K[\mathcal{A}_T]$  with  $\emptyset \neq T \subseteq [n]$  is called a combinatorial pure subring of  $K[\mathcal{A}]$ . For  $\mathcal{A}_T = \{u_{i_1}, \dots, u_{i_r}\}$ , we set  $S_{\mathcal{A}_T} = \{y_{u_{i_1}}, \dots, y_{u_{i_r}}\}$ . Therefore  $I_{\mathcal{A}_T} = I_{\mathcal{A}} \cap S_{\mathcal{A}_T}$ .

**Lemma 2.7.** *Let  $\alpha \succeq \beta$  be two  $n$ -tuple vectors in  $\mathbb{N}^n$ . Then  $K[\mathcal{A}^\beta]$  is a combinatorial pure subring of  $K[\mathcal{A}^\alpha]$ .*

*Proof.* Let  $\alpha = (k_1, \dots, k_n)$  and  $\beta = (l_1, \dots, l_n)$ . Set  $T = \{x_{11}, \dots, x_{1l_1}, \dots, x_{n1}, \dots, x_{nl_n}\}$ . It is clear that  $(\mathcal{A}^\alpha)_T = \mathcal{A}^\beta$ .  $\square$

**Remark 2.8.** In [11] the authors showed that if  $\mathcal{A}$  is a homogeneous affine semigroup ring generated by monomials belonging to a polynomial ring with the toric ideal  $I$  which is normal, Golod, Koszul, strongly Koszul, sequentially Koszul or extendable sequentially Koszul, then any of its combinatorial pure subrings, as  $B$ , inherits each of these properties. Moreover, if  $\mathcal{G}$  is any reduced Gröbner basis of  $I$  then  $\mathcal{G} \cap B$  is the reduced Gröbner basis of  $J$ , where  $J$  is the toric ideal of  $B$ .

Lemma 2.7 implies that  $K[\mathcal{A}]$  is a combinatorial pure subring of  $K[\mathcal{A}^\alpha]$  and so if  $K[\mathcal{A}^\alpha]$  has one of the above properties, then  $K[\mathcal{A}]$  has the same property, too.

In the following we investigate some algebraic properties for  $K[\mathcal{A}^\alpha]$  when they hold for  $K[\mathcal{A}]$ .

## Gröbner basis

Proposition 1.1 of [11] together with Lemma 2.7 guarantees the following:

**Proposition 2.9.** *Let  $\alpha \in \mathbb{N}^n$  and let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ . If  $\mathcal{G}$  is the reduced Gröbner basis of  $I_{\mathcal{A}^\alpha}$  with respect to a term order  $<$  on  $S_{\mathcal{A}^\alpha}$ , then  $\mathcal{G} \cap K[\mathcal{A}]$  is the reduced Gröbner basis of  $I_{\mathcal{A}}$  with respect to a term order induced by  $<$  on  $S_{\mathcal{A}}$ .*

Let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ . Define the term order “ $<_{\text{lex}}^\#$ ” on the variables of  $\{y_{u_1}, \dots, y_{u_m}\}$  in the following form:

$$y_u <_{\text{lex}}^\# y_v \Leftrightarrow u <_{\text{lex}} v \text{ and } y_u = y_v \Leftrightarrow u = v.$$

Also, consider the ordering  $<_{\text{Lex}}$  induced by

$$x_{11} > \dots > x_{1k_1} > \dots > x_{n1} > \dots > x_{nk_n}$$

on the monomials of  $\mathcal{A}^\alpha$  for  $\alpha = (k_1, \dots, k_n)$ . Again, let “ $<_{\text{Lex}}^\#$ ” be a term order on the variables of  $\{y_{u'} : u' \in \mathcal{A}^\alpha\}$  in the following form:

$$y_{u'} <_{\text{Lex}}^\# y_{v'} \Leftrightarrow u' <_{\text{Lex}} v' \text{ and } y_{u'} = y_{v'} \Leftrightarrow u' = v'.$$

**Lemma 2.10.** Let  $\mathcal{A}$  be a finite set of monomials belonging to  $S = K[x_1, \dots, x_n]$  and let  $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ . For  $\beta = \alpha + \varepsilon_i$  there exists a  $K$ -algebra isomorphism

$$\varphi : K[(\mathcal{A}^\alpha)^\gamma] \rightarrow K[\mathcal{A}^\beta]$$

where  $\gamma = \mathbf{1} + \varepsilon_{ik_i} \in \mathbb{N}^{|\alpha|}$ . Here  $\mathbf{1}$  is the vector in  $\mathbb{N}^{|\alpha|}$  with all components 1.

*Proof.* For  $\alpha$  we have  $[n]^\alpha = \{x_{11}, \dots, x_{1k_1}, \dots, x_{n1}, \dots, x_{nk_n}\}$ . Also,

$$([n]^\alpha)^\gamma = \{x_{111}, \dots, x_{1k_1 1}, \dots, x_{(i-1)k_{i-1} 1}, x_{ik_i 1}, x_{ik_i 2}, x_{(i+1)k_{i+1} 1}, \dots, x_{n11}, \dots, x_{nk_n 1}\}.$$

Consider the relabeling  $\sigma : ([n]^\alpha)^\gamma \rightarrow [n]^\beta$  by

$$\sigma(x_{rst}) = \begin{cases} x_{rs} & \text{if } t = 1 \\ x_{i(k_i+1)} & \text{if } t = 2. \end{cases}$$

Then the  $K$ -algebra homomorphism

$$\varphi : K[(\mathcal{A}^\alpha)^\gamma] \rightarrow K[\mathcal{A}^\beta]$$

defined by  $\varphi(u) = \prod_{x_{rst} | u} \sigma(x_{rst})$  for all monomials  $u \in (\mathcal{A}^\alpha)^\gamma$  is an isomorphism.  $\square$

**Theorem 2.11.** Let  $\mathcal{A}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$  and let  $\alpha = \mathbf{1} + \varepsilon_i \in \mathbb{N}^n$ . If  $\mathcal{G}_\mathcal{A}$  is a Gröbner basis of  $I_\mathcal{A}$  with respect to a term order induced by  $<_{\text{lex}}^\#$  on  $S_\mathcal{A}$ , then the Gröbner basis of  $I_{\mathcal{A}^\alpha}$ ,  $\mathcal{G}_{\mathcal{A}^\alpha}$ , with respect to the term order induced by  $<_{\text{Lex}}^\#$  on  $S_{\mathcal{A}^\alpha}$  is the union of the set

$$\mathcal{G}_0 := \{y_{u'} y_{v'} - y_{(u'/x_{i1})x_{i2}} y_{(v'/x_{i2})x_{i1}} : u', v' \in \mathcal{A}^\alpha \text{ and } x_{i1} | u', x_{i2} | v'\}$$

and the set, call  $\mathcal{G}_1$ , containing all binomials  $\prod_l^r y_{u'_l} - \prod_l^s y_{v'_l}$  with the property that  $\prod_l^r y_{\pi(u'_l)} - \prod_l^s y_{\pi(v'_l)} \in \mathcal{G}_\mathcal{A}$  and  $\prod_l^r u'_l = \prod_l^s v'_l$  for  $u'_l, v'_l \in \mathcal{A}^\alpha$ .

*Proof.* It is known that toric ideals are binomial. To showing that  $\mathcal{G}_0 \cup \mathcal{G}_1$  is a Gröbner basis of  $I_{\mathcal{A}^\alpha}$ , suppose  $0 \neq f' = \prod_l^r y_{u'_l} - \prod_l^s y_{v'_l} \in I_{\mathcal{A}^\alpha}$  and for all  $l$  and  $m$  with  $l \neq m$ ,  $u'_l \neq v'_m$ . We want to show that  $\text{in}_{<_{\text{Lex}}^\#}^\#(f')$  is divided by  $\text{in}_{<_{\text{Lex}}^\#}^\#(g')$  for some  $g' \in \mathcal{G}_0 \cup \mathcal{G}_1$ . Let  $f := \gamma(f') \in I_\mathcal{A}$ . We have two cases:

**Case 1.** Assume that  $f = 0$ . Then  $r = s$  and we can assume that  $\pi(u'_l) = \pi(v'_l)$  for all  $l$ .

If  $x_i \nmid \pi(u'_l)$ , for some  $l$ , then  $u'_l = v'_l$  which contradicts the assumption. Thus  $x_i | \pi(u'_l)$  for all  $l$ .

Assume that  $\text{in}_{<_{\text{Lex}}^\#}^\#(f') = \prod_l^r y_{u'_l}$ . Note that there are two distinct indexes  $l_1$  and  $l_2$  such that  $x_{i1} | u'_{l_1}$  and  $x_{i2} | u'_{l_2}$ . Otherwise, we will have  $u'_l = v'_l$  for all  $l$ , which is not true. Suppose that  $u'_r \leq_{\text{Lex}} \dots \leq_{\text{Lex}} u'_2 \leq_{\text{Lex}} u'_1$  and  $l_1 = 1$ . Furthermore, we can assume that  $u'_{l_2}$  is a monomial such that  $v_{x_{i2}}(v'_{l_2}) < v_{x_{i2}}(u'_{l_2})$ . Otherwise, since  $v_{x_{i2}}(v'_1) > v_{x_{i2}}(u'_1)$ , it follows that  $v_{x_{i2}}(\prod_l^r u'_l) < v_{x_{i2}}(\prod_l^s v'_l)$ , which is a contradiction.

Suppose that  $l_2 = 2$ . Set  $h := y_{u'_1} y_{u'_2} - y_{(u'_1/x_{i1})x_{i2}} y_{(u'_2/x_{i2})x_{i1}}$ . Then  $h \in I_{\mathcal{A}^\alpha}$ . It is clear that  $(u'_1/x_{i1})x_{i2} <_{\text{Lex}} u'_1$ . Also, if  $u'_1 \leq_{\text{Lex}} (u'_2/x_{i2})x_{i1}$ , then using  $(u'_2/x_{i2})x_{i1} \leq_{\text{Lex}} v'_2$  we obtain  $u'_1 \leq_{\text{Lex}} v'_2$  and so  $\text{in}_{<_{\text{Lex}}^\#}^\#(f') = \prod_l^r y_{v'_l}$ , a contradiction. Therefore  $u'_1 >_{\text{Lex}} (u'_2/x_{i2})x_{i1}$ . This implies that  $\text{in}_{<_{\text{Lex}}^\#}^\#(h) = y_{u'_1} y_{u'_2}$ . In particular,  $\text{in}_{<_{\text{Lex}}^\#}^\#(h) | \text{in}_{<_{\text{Lex}}^\#}^\#(f')$  and  $h \in \mathcal{G}_0$ .

**Case 2.** Assume that  $f \neq 0$ . Let  $\pi(u'_l) = u_l$ ,  $\pi(v'_l) = v_l$  and  $\text{in}_{<_{\text{Lex}}^\#}^\#(f) = \prod_l^r y_{u_l}$ . Since  $f \in I_\mathcal{A}$ , there exists  $g = \prod_l^p y_{u_l} - \prod_l^q y_{v_l} \in \mathcal{G}_\mathcal{A}$  with  $\text{in}_{<_{\text{Lex}}^\#}^\#(g) = \prod_l^p y_{u_l} | \text{in}_{<_{\text{Lex}}^\#}^\#(f)$ .

If  $\text{in}_{<_{\text{Lex}}}^{\#}(f') = \prod_l^r y_{u'_l}$ , then we can choose some monomials  $w'_l$  from  $\mathcal{A}^\alpha$ , with the property that  $\pi(w'_l) = w_l$ ,  $\prod_l^p u'_l = \prod_l^q w'_l$  and  $\text{in}_{<_{\text{Lex}}}^{\#}(g') = \prod_l^p y_{u'_l}$ . Set  $g' := \prod_l^p y_{u'_l} - \prod_l^q y_{w'_l}$ . Then  $\text{in}_{<_{\text{Lex}}}^{\#}(g') | \text{in}_{<_{\text{Lex}}}^{\#}(f')$  and  $g' \in \mathcal{G}_1$ .

If  $\text{in}_{<_{\text{Lex}}}^{\#}(f') = \prod_l^s y_{v'_l}$ , then it follows from  $\text{in}_{<_{\text{Lex}}}^{\#}(f) = \prod_l^r y_{u_l}$  that there are two (possibly equal) monomials  $v'_h$  and  $v'_k$  such that  $x_{i1} | v'_h$  and  $x_{i2} | v'_k$ . More precisely, let  $v'_s \leq_{\text{Lex}} \dots \leq_{\text{Lex}} v'_2 \leq_{\text{Lex}} v'_1$  and  $u_r \leq_{\text{lex}} \dots \leq_{\text{lex}} u_2 \leq_{\text{lex}} u_1$ . Since  $u_1 \geq_{\text{lex}} v_1$  and  $v'_1 \geq_{\text{Lex}} u'_1$ , it is clear that  $x_i$  divides both  $u_1$  and  $v_1$ . If for some  $j < i$ ,  $v_{x_j}(u_1) > v_{x_j}(v_1)$ , then  $u'_1 >_{\text{Lex}} v'_1$ , which is false. Thus suppose that for all  $j < i$ ,  $v_{x_j}(u_1) = v_{x_j}(v_1)$ . Also, it is easily to verify that  $v_{x_i}(u_1) \geq v_{x_i}(v_1)$ ,  $v_{x_{i1}}(v'_1) > v_{x_{i1}}(u'_1)$  and  $v_{x_{i2}}(v'_1) < v_{x_{i2}}(u'_1)$ . Especially, since  $\prod_l^r u'_l = \prod_l^s v'_l$ , it follows that there is some monomial, call  $v'_k$ , such that  $x_{i2} | v'_k$ .

If  $v'_k$  has a property that  $v'_1 \geq_{\text{Lex}} x_{i1}(v'_k/x_{i2})$ , then by setting  $f'' := y_{v'_1} y_{v'_k} - y_{x_{i2}(v'_1/x_{i1})} y_{x_{i1}(v'_k/x_{i2})}$  we will have  $\text{in}_{<_{\text{Lex}}}^{\#}(f'') | \text{in}_{<_{\text{Lex}}}^{\#}(f')$ . Since  $f'' \in \mathcal{G}_0$ , the assertion is completed. Hence assume that for every  $v'_l$  with  $l \geq 2$ , if  $x_{i2} | v'_l$  then  $v'_1 <_{\text{Lex}} x_{i1}(v'_l/x_{i2})$ . In particular, for such  $v'_l$ 's, we will have  $v_{x_{i1}}(v'_l) = v_{x_{i1}}(v'_1) - 1$  and  $v_{x_{i2}}(v'_l) \geq v_{x_{i2}}(v'_1) + 1$ .

Since  $f \in I_{\mathcal{A}}$ , there exists  $g = y_{u_1} \dots y_{u_p} - y_{w_1} \dots y_{w_q} \in \mathcal{G}_{\mathcal{A}}$  such that  $\text{in}_{<_{\text{Lex}}}^{\#}(g) = y_{u_1} \dots y_{u_p} | \text{in}_{<_{\text{Lex}}}^{\#}(f)$ . Set  $h := \prod_l^s y_{v_l} - (y_{w_1} \dots y_{w_q} \prod_{l \neq t_i} y_{u_l})$ .

- (1) If  $\prod_l^s y_{v_l} = y_{w_1} \dots y_{w_q} \prod_{l \neq t_i} y_{u_l}$  then, by assuming  $w_1 = v_{s_1}, \dots, w_q = v_{s_q}$  and setting  $g' := y_{u'_1} \dots y_{u'_p} - y_{v'_{s_1}} \dots y_{v'_{s_q}}$ , we will have  $\text{in}_{<_{\text{Lex}}}^{\#}(g') | \text{in}_{<_{\text{Lex}}}^{\#}(f')$  and especially  $g' \in \mathcal{G}_1$ .
- (2) If  $\prod_l^s y_{v_l} >_{\text{Lex}} y_{w_1} \dots y_{w_q} \prod_{l \neq t_i} y_{u_l}$ , then set  $h' = \prod_l^s y_{v'_l} - (y_{w'_1} \dots y_{w'_q} \prod_{l \neq t_i} y_{u'_l})$ . Thus there exists  $g_0 = y_{v_{s_1}} \dots y_{v_{s_p}} - y_{z_1} \dots y_{z_k}$  such that  $\text{in}_{<_{\text{Lex}}}^{\#}(g_0) = y_{v_{s_1}} \dots y_{v_{s_p}} | \text{in}_{<_{\text{Lex}}}^{\#}(h) = \text{in}_{<_{\text{Lex}}}^{\#}(f)$ . Again set  $g'_0 = y_{v'_{s_1}} \dots y_{v'_{s_p}} - y_{z'_1} \dots y_{z'_k}$ . Now  $\text{in}_{<_{\text{Lex}}}^{\#}(g'_0) = y_{v'_{s_1}} \dots y_{v'_{s_p}} | \text{in}_{<_{\text{Lex}}}^{\#}(f')$  and  $g'_0 \in \mathcal{G}_1$ .
- (3) If  $\prod_l^s y_{v_l} <_{\text{Lex}} y_{w_1} \dots y_{w_q} \prod_{l \neq t_i} y_{u_l}$ , then there is  $g_1 = y_{w_{h_1}} \dots y_{w_{h_k}} \prod_{l \neq t_i} y_{u_{l_j}} - y_{z_1} \dots y_{z_s} \in \mathcal{G}_{\mathcal{A}}$  such that  $\text{in}_{<_{\text{Lex}}}^{\#}(g_1) = y_{w_{h_1}} \dots y_{w_{h_k}} \prod_{l \neq t_i} y_{u_{l_j}} | \text{in}_{<_{\text{Lex}}}^{\#}(h)$ . Set

$$h_1 := \prod_l^s y_{v_l} - (y_{z_1} \dots y_{z_s} \prod_{l \neq l_j, l \neq t_i} y_{u_l} \prod_{l \neq h_i} y_{w_l}).$$

Now if  $\prod_l^s y_{v_l} \geq_{\text{Lex}} y_{z_1} \dots y_{z_s} \prod_{l \neq l_j, l \neq t_i} y_{u_l} \prod_{l \neq h_i} y_{w_l}$ , then by (1) and (2) the assertion is completed. Otherwise, we again go back to (3).

Using the above procedure, after only finitely many steps, we obtain  $g_k = \prod_l^a y_{v_{l_i}} - \prod_l^b y_{x_l} \in \mathcal{G}_{\mathcal{A}}$  with  $\text{in}_{<_{\text{Lex}}}^{\#}(g_k) = \prod_l^a y_{v_{l_i}}$ . Now set  $g'_k := \prod_l^a y_{v'_{l_i}} - \prod_l^b y_{x'_l}$ . Then  $\text{in}_{<_{\text{Lex}}}^{\#}(g'_k) | \text{in}_{<_{\text{Lex}}}^{\#}(f')$  and  $g'_k \in \mathcal{G}_1$ , as desired.  $\square$

**Example 2.12.** Consider a set

$$\mathcal{A} = \{x_1^2 x_2, x_1 x_2 x_4, x_1 x_3, x_2 x_3^2, x_2 x_4^2\}$$

of monomials belonging to  $K[x_1, \dots, x_4]$ . Using CoCoA we obtain

$$\mathcal{G}_{\mathcal{A}} = \{y_{x_1^2 x_2} y_{x_2 x_4^2} - y_{x_1 x_2 x_4}^2\}.$$

Let  $\alpha = (1, 1, 1, 2)$ . With the same notations of Theorem 2.11 we have  $\mathcal{G}_{\mathcal{A}^\alpha} = \mathcal{G}_0 \cup \mathcal{G}_1$  where

$$\mathcal{G}_0 = \{y_{x_1 x_2 x_4} y_{x_2 x_4 x_4} - y_{x_1 x_2 x_4} y_{x_2 x_4^2}, y_{x_2 x_4^2} y_{x_2 x_4^2} - y_{x_2 x_4 x_4}^2, y_{x_1 x_2 x_4} y_{x_2 x_4^2} - y_{x_1 x_2 x_4} y_{x_2 x_4 x_4}\}$$

and

$$\mathcal{G}_1 = \{y_{x_1^2 x_2} y_{x_2 x_{41}^2} - y_{x_1 x_2 x_{41}}^2, y_{x_1^2 x_2} y_{x_2 x_{42}^2} - y_{x_1 x_2 x_{42}}^2, y_{x_1^2 x_2} y_{x_2 x_{41} x_{42}} - y_{x_1 x_2 x_{41}} y_{x_1 x_2 x_{42}}\}.$$

Combining Proposition 2.9, Lemma 2.10 and Theorem 2.11 we have the following.

**Corollary 2.13.** *For a given  $\alpha \in \mathbb{N}^n$  and a set  $\mathcal{A}$  of monomials in  $S$ , the Gröbner basis of  $I_{\mathcal{A}}$  consists of quadratic binomials if and only if the Gröbner basis of  $I_{\mathcal{A}^\alpha}$  has a such construction.*

## Sortable sets

The concept of sortable sets was introduced by Sturmfels [13]. Let  $\mathcal{A}$  be a set of monomials in  $S$  and let  $u, v \in \mathcal{A}$ . Write  $uv = x_{i_1} \dots x_{i_{2d}}$  with  $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_{2d}}$ . Set  $u' = \prod_j^d x_{2j-1}$  and  $v' = \prod_j^d x_{2j}$ . We define

$$\text{sort} : \mathcal{A} \times \mathcal{A} \rightarrow M_d \times M_d$$

with  $\text{sort}((u, v)) = (u', v')$  where  $M_d$  denotes the set of all vectors in  $\mathbb{Z}^n$  of modulus  $d$ .  $\mathcal{A}$  is called *sortable*, if  $\text{im}(\text{sort}) \subseteq \mathcal{A} \times \mathcal{A}$ .

For  $\alpha \in \mathbb{N}^n$ , we denote by  $M_d^\alpha$  the set of all vectors in  $\mathbb{Z}^{|\alpha|}$  of modulus  $d$ . We define

$$\text{sort}^\alpha : \mathcal{A}^\alpha \times \mathcal{A}^\alpha \rightarrow M_d^\alpha \times M_d^\alpha$$

with  $\text{sort}^\alpha((u, v)) = (u', v')$

**Theorem 2.14.** *Let  $\alpha \in \mathbb{N}^n$ . Let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ .  $\mathcal{A}$  is a sortable set if and only if  $\mathcal{A}^\alpha$  has a such property.*

*Proof.* Let  $(u', v') \in \text{im}(\text{sort}^\alpha)$ . Then there exists  $(u, v) \in \mathcal{A}^\alpha \times \mathcal{A}^\alpha$  such that  $\text{sort}^\alpha((u, v)) = (u', v')$ . It is easy to see that  $\text{sort}((\pi(u), \pi(v))) = (\pi(u'), \pi(v'))$ . Now since  $\mathcal{A}$  is sortable, we have that  $(\pi(u'), \pi(v')) \in \mathcal{A} \times \mathcal{A}$ . Therefore  $(u', v') \in \mathcal{A}^\alpha \times \mathcal{A}^\alpha$ .

For the converse direction, suppose that  $(u'_0, v'_0) \in \text{im}(\text{sort})$ . Then there exists  $(u_0, v_0) \in \mathcal{A} \times \mathcal{A}$  such that  $\text{sort}((u_0, v_0)) = (u'_0, v'_0)$ . Set  $u = \prod_{x_i|u_0} x_{1i}$  and  $v = \prod_{x_i|v_0} x_{1i}$ . It is clear that  $\text{sort}^\alpha((u, v)) = (u', v')$  which  $u' = \prod_{x_i|u'_{1i}} x_{1i}$  and  $v' = \prod_{x_i|v'_{1i}} x_{1i}$ . Now since  $\mathcal{A}^\alpha$  is sortable, so  $(u', v') \in \mathcal{A}^\alpha \times \mathcal{A}^\alpha$ . This implies that  $(u'_0, v'_0) \in \mathcal{A} \times \mathcal{A}$ , as desired.  $\square$

By a result due to Sturmfels [13], toric ideal associated to a sortable set  $\mathcal{A}$  has a Gröbner basis consisting of the sorting relations  $y_u y_v - y_{u'} y_{v'}$  with  $u, v \in \mathcal{A}$  and  $(u', v') = \text{sort}((u, v))$ . This result together with Theorem 2.14 follows that:

**Corollary 2.15.** *Let  $\alpha \in \mathbb{N}^n$  and let  $\mathcal{A}$  be a set of monomials belonging to  $S$ . If  $\mathcal{A}$  (resp.  $\mathcal{A}^\alpha$ ) is sortable then  $I_{\mathcal{A}^\alpha}$  (resp.  $I_{\mathcal{A}}$ ) has a Gröbner basis consisting of the quadratic sorting relations.*

## Normalness

For  $\mathcal{A} = \{\mathbf{x}^{\mathbf{u}_1}, \dots, \mathbf{x}^{\mathbf{u}_m}\}$  a set of monomials belonging to  $S$ , we set  $\vec{\mathcal{A}} := \log_{\mathbf{x}}(\mathcal{A}) = \{\mathbf{u}_1, \dots, \mathbf{u}_m\} \subset \mathbb{Z}_+^n$ .

**Theorem 2.16.** *Let  $\alpha \in \mathbb{N}^n$ . Let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ . Then  $K[\mathcal{A}]$  is a normal ring if and only if  $K[\mathcal{A}^\alpha]$  has this property.*

*Proof.* By Theorem 6.1.4 of [3],  $K[\mathcal{A}]$  is normal if and only if  $\mathcal{A}$  is a normal affine semigroup, i.e. if  $t\mathbf{u} \in \mathcal{A}$  for some  $\mathbf{u} \in \mathbb{Z}\mathcal{A}$  and  $t \in \mathbb{N}$ , then  $\mathbf{u} \in \mathcal{A}$ . Here  $\mathbb{Z}\mathcal{A}$  means a smallest group containing  $\mathcal{A}$ . Thus it suffices to show that  $(\mathcal{A})^\alpha$  is a normal semigroup when  $\mathcal{A}$  is normal.

“Only if part”: Let  $t\mathbf{u} \in (\mathcal{A})^\alpha$  for some  $\mathbf{u} \in \mathbb{Z}(\mathcal{A})^\alpha$  and  $t \in \mathbb{N}$ . It follows that  $t\pi_0(\mathbf{u}) \in \mathcal{A}$  and  $\pi_0(\mathbf{u}) \in \mathbb{Z}\mathcal{A}$ . Let  $\mathbf{u} = (u_{11}, \dots, u_{1k_1}, \dots, u_{n1}, \dots, u_{nk_n})$ . Since  $\mathcal{A}$  is normal, we conclude that  $\pi_0(\mathbf{u}) \in \mathcal{A}$ . Let  $\pi_0(\mathbf{u}) = (u_1, \dots, u_n)$ . Since  $\mathbf{u} \in \mathbb{Z}(\mathcal{A})^\alpha$ , every component of  $\mathbf{u}$  is integer. Now  $u_i = \sum_{j=1}^{k_i} u_{ij}$  implies that  $\mathbf{u} \in (\mathcal{A})^\alpha$ , as desired.

“If part”: Let  $t\mathbf{u} \in \mathcal{A}$  for some  $\mathbf{u} \in \mathbb{Z}\mathcal{A}$  and  $t \in \mathbb{N}$ . Set  $\mathbf{u}' := \sum_i^n \mathbf{u}(i)\varepsilon_{i1}$ . It is clear that  $t\mathbf{u}' \in (\mathcal{A})^\alpha$  and so  $\mathbf{u}' \in \mathbb{Z}(\mathcal{A})^\alpha$ . In particular,  $\mathbf{u} = \pi_0(\mathbf{u}') \in \mathcal{A}$ .  $\square$

**Remark 2.17.** The if part of Theorem 2.16 is a straightforward consequence of Proposition 1.2 of [11] and Lemma 2.7.

## Some homological relations

Corollary 2.5 of [11] together with Lemma 2.7 obtain the following result.

**Proposition 2.18.** Let  $\alpha \in \mathbb{N}^n$  and let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ . For the graded Betti numbers of  $I_{\mathcal{A}^\alpha}$  and  $I_{\mathcal{A}}$  we have

$$\beta_{ij}^{K[\mathcal{A}]}(I_{\mathcal{A}}) \leq \beta_{ij}^{K[\mathcal{A}^\alpha]}(I_{\mathcal{A}^\alpha}) \quad \text{for all } i \text{ and } j.$$

**Theorem 2.19.** Let  $\alpha \in \mathbb{N}^n$  and let  $\mathcal{A} = \{u_1, \dots, u_m\}$  be a set of monomials belonging to  $S = K[x_1, \dots, x_n]$ . Then the following hold:

- (a)  $\dim(K[\mathcal{A}]) \leq \dim(K[\mathcal{A}^\alpha])$ ;
- (b)  $\text{depth}(K[\mathcal{A}]) \leq \text{depth}(K[\mathcal{A}^\alpha])$ ;
- (c)  $\text{proj.dim}(K[\mathcal{A}]) \leq \text{proj.dim}(K[\mathcal{A}^\alpha])$ ;
- (d)  $\text{reg}(K[\mathcal{A}]) \leq \text{reg}(K[\mathcal{A}^\alpha])$ .

*Proof.* (a) It is easily seen that  $\text{rank}(\mathcal{A}) \leq \text{rank}(\mathcal{A}^\alpha)$ . On the other hand, it is known that  $\dim(K[\mathcal{A}]) = \text{rank}(\mathcal{A})$  (See [3, Chapter 6]). Therefore the desired equality follows immediately.

(b) It is a straightforward consequence of Proposition 2.4 of [11], Theorem A.3.4 of [6] and Lemma 2.7.

(c) and (d) follows from Proposition 2.18.  $\square$

**Remark 2.20.** One may ask that for a set  $\mathcal{A}$  of monomials if the toric ring  $K[\mathcal{A}]$  is Cohen-Macaulay, is  $K[\mathcal{A}^\alpha]$  Cohen-Macaulay, too? The answer is negative. For instance, consider  $\mathcal{A} = \{x_1^3, x_1^2x_2, x_2^3\} \subset K[x_1, x_2]$  and  $\alpha = (2, 2) \in \mathbb{N}^2$ . By using CoCoA, we see that  $K[\mathcal{A}]$  is a Cohen-Macaulay ring but  $K[\mathcal{A}^\alpha]$  is not.  $K[\mathcal{A}]$  is of dimension 2, while the dimension and depth of  $K[\mathcal{A}^\alpha]$  are, respectively, 4 and 3.

## Acknowledgment

The authors would like to thank the referee for a careful reading of this note and for valuable comments and corrections. The research of Rahim Rahmati-Asghar was in part supported by a grant from IPM (No. 92130029). The research of Siamak Yassemi was in part supported by a grant from the University of Tehran (No. 6103023/1/014).

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